Problem 19

If $f_0(x) = x^2$ and $f_{n+1}(x) = f_0(f_n(x))$ for $n = 0, 1, 2, \ldots$, find a formula for $f_n(x)$.

Solution

Write out the first few formulas using the given definitions and try to find a pattern.

$$\begin{aligned}
f_0(x) &= x^2 \\
f_1(x) &= f_{0+1}(x) = f_0(f_0(x)) = (x^2)^2 = x^4 \\
f_2(x) &= f_{1+1}(x) = f_0(f_1(x)) = (x^4)^2 = x^8 \\
f_3(x) &= f_{2+1}(x) = f_0(f_2(x)) = (x^8)^2 = x^{16} \\
f_4(x) &= f_{3+1}(x) = f_0(f_3(x)) = (x^{16})^2 = x^{32} \\
&\vdots \\
f_n(x) &= f_{(n-1)+1}(x) = f_0(f_{n-1}(x)) = (x^{2^n})^2 = x^{2(2^n)} = x^{2^{n+1}}
\end{aligned}$$